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COMMENT

The equivalent Schrödinger equation for the bounce solution

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Abstract. We study the equivalent Schrödinger equation for the bounce configuration. We get the lowest eigenvalue in the limit of small external currents using the Landau-Lifshitz approach.

1. Introduction

The path integral formulation for quantum models is limited by the fact that we are only able to integrate Gaussians. In quantum mechanics and in quantum field theory, where we have interactions (polynomial potentials whose highest power is bigger than two), in general we can obtain non-perturbative information by using the semi-classical approximation [1, 2]. In the semi-classical approximation, we calculate the Gaussian correction to classical configurations that are solutions of the Euler-Lagrange equation [2] that satisfy the boundary conditions of the path integral. This Gaussian correction reduces in general to discuss the spectrum of an associated Schrödinger equation with a given potential—the equivalent Schrödinger problem.

For theories that classically have degenerate minima of the potential, e.g. $\vartheta(x) = g/4(x^2-a^2)^2$, the solutions of the Euler-Lagrange equation are the constant configurations $x = \pm a$, or the kink solutions [1], depending on the boundary conditions. The semi-classical contribution from these configurations have been studied in the literature [3, 4].

The aim of this comment is to derive the lowest energy eigenvalue of the Schrödinger equivalent problem associated with the semi-classical approximation from the bounce solution for a one-dimensional theory with a potential of the form $\vartheta(x) = (g/4)(x^2-a^2)^2 - Jx$, where $x = x(t)$ is a time-dependent coordinate and J an external current, for small values of J. In this case, the linear term in x breaks the degeneracy and, for boundary conditions such that $\lim_{t\to\pm\infty} x(t) = \pm a$, we have the so-called bounce solution [5]. The Schrödinger equivalent operator has a negative eigenvalue since J is different from zero. The fact that $J \to 0^+$ but J is not exactly zero also means that the classical potential is almost degenerate.

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In section 2 we review the Schrödinger equivalent problem for the kink solution. In section 3 we discuss the Schrödinger equivalent operator when $J \to 0^+$, from the exact expression for the bounce configuration, and in section 4 we discuss some features of the problem. In the appendix we give the integrals needed to get the final result.

2. Review of the Gaussian contribution to the kink solution

The double-well potential $V(x)=\frac{1}{4}g(x^2-a^2)^2$ has degenerate minima at $x=\pm a$, at the classical level. However, at the quantum level, this degeneracy is lifted by the existence of a solution (the kink) that links the two degenerate minima [1, 3]. The kink is an extremum of the Euclidean action (the Euclidean action is obtained by going to imaginary time, $t \to \mathrm{i}t$. For the double-well potential, the Euclidean Lagrangian for unit mass is

$$L_E = -\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \frac{g}{4}(x^2-a^2)^2 \tag{2.1}$$

and the solution of the Euler–Lagrange equation satisfying the conditions that for $t \to -\infty$, $x=-a$ and $t \to +\infty$, $x=+a$ is

$$\bar{x}(t) = a \tanh\left(\sqrt{\frac{ga^2}{2}}(t-t_0)\right) \tag{2.2}$$

which is known as the kink solution.

The Gaussian contribution from the kink configuration is obtained considering a fluctuation around $\bar{x}(t)$,

$$x(t)=\bar{x}(t)+\eta(t) \tag{2.3}$$

and keeping only terms up to second order in the fluctuation $\eta(t)$. Then the Euclidean kernel $\langle a, +\infty|-a, -\infty\rangle$, which allows the calculation of the lowest energy level and the splitting of the two first levels, is given by the path integral

$$Z = \exp(-S[\bar{x}]) \int \mathscr{D}x \exp\left(-\int \mathrm{d}t\, \eta(t) H_0 \eta(t)\right) \tag{2.4}$$

where $S[\bar{x}]$ is the classical action associated with the kink,

$$S[\bar{x}] = \int \mathrm{d}t\left(\frac{\dot{\bar{x}}^2}{2} + V(\bar{x})\right) \tag{2.5}$$

and H_0 is the operator,

$$H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + ga^2\left[1-\frac{3}{2}\operatorname{sech}^2\left(\sqrt{\frac{ga^2}{2}}\,t\right)\right]. \tag{2.6}$$

In (2.4)–(2.6) the integrations over t run from $-T$ to $+T$ and the limit $T \to \infty$ is understood to be taken at the end of the calculations.

The operator H_0 is a Schrödinger-like Hamiltonian operator for a unit-mass particle. Their eigenfunctions and eigenvalues are well-known energy eigenfunctions and eigenvalues [6].

3. The Schrödinger operator in the bounce case

The system whose classical potential is

$$\vartheta(x)=\frac{g}{4}(x^2-a^2)^2-Jx \tag{3.1}$$

may be seen as the system described in section 2 in the presence of an external current J. For $J\neq 0$, $\vartheta(x)$ does not have degenerate minima (see figure 1), and we are interested in obtaining information when $J\to 0^+$ and we are in Euclidean space.

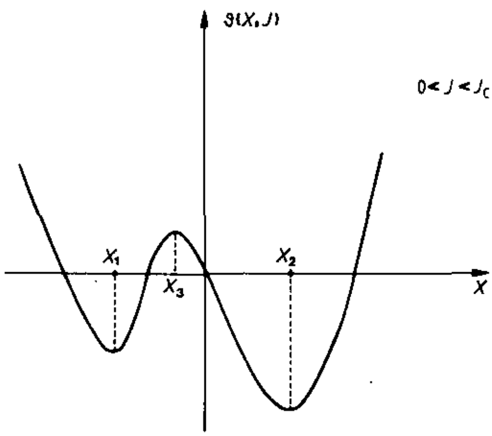

Figure 1. The shape of the potential $\vartheta(x)$ for $J\in(0,J_c)$. x_1 and x_2 are the relative and absolute minima of $\vartheta(x)$ respectively and x_3 its relative maximum.

The bounce is a solution of the Euclidean equation of motion which has finite energy, and is such that starting in the remote 'past' in the local (non-absolute) minimum of $\vartheta(x)$ (see figure 1), suffers reflection at the potential barrier associated with the other minimum and returns in the remote 'future' to the original one. The exact analytic expression of the bounce is [7]

$$\bar{x}_B(t)=x_1+\frac{1}{2}\sqrt{\frac{4A}{g}}\left[\tanh\left(\sqrt{\frac{A}{2}}(t+\bar{t}-t_0)\right)-\tanh\left(\sqrt{\frac{A}{2}}(t-\bar{t}-t_0)\right)\right] \tag{3.2}$$

where t_0 is an arbitrary constant that appears as a consequence of time translation invariance of the Euclidean Lagrangian, A and $\bar{t}$ are constants,

$$A=\frac{g}{2}(3x_1^2-a^2) \tag{3.3a}$$

$$\bar{t}=\frac{1}{\sqrt{2A}}\tanh^{-1}\left(\frac{\sqrt{gA}}{gx_1}\right) \tag{3.3b}$$

and x_1 is the position of the relative minimum of $\vartheta(x)$ as indicated in figure 1. We note that $\lim_{t\to\pm\infty}\bar{x}(t)=x_1$. From now on we take $t_0=0$.

The contribution from the fluctuation around $\bar{x}_B$ to the path integral is calculated in the same way as we did for the symmetric double-well potential in section 2, by making a change of variables of the type (2.3) around the bounce solution $\bar{x}_B(t)$.

The path integral is given by

$$Z(J) = \exp(-S[\bar{x}_B]) \exp\left\{-\int \mathrm{d}t\left[\eta\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}+U(t)\right)\eta + g\bar{x}_B\eta^3 + \frac{g}{4}\eta^4\right]\right\} \tag{3.4}$$

where $S[\bar{x}_B]$ is the bounce classical action,

$$S[\bar{x}_B] = \int \mathrm{d}t[\tfrac{1}{2}(\dot{\bar{x}}_B)^2 + \vartheta(\bar{x}_B)] \tag{3.5}$$

and $U(t)$ is a double-well potential function whose shape is illustrated in figure 2,

$$U(t) = A\left\{1 - \frac{3}{2}\left[\operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t+\bar{t})\right) + \operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t-\bar{t})\right)\right]\right\}. \tag{3.6}$$

The separation between the wells is $2\bar{t}$, with, for small values of J,

$$\bar{t} \approx -\frac{1}{2\sqrt{2g}}\frac{1}{a}\ln\left(\frac{3\sqrt{3}J}{2ga^3}\right). \tag{3.7}$$

The distance between the wells increases as J goes to zero. We should notice that for $J = 0$, we have $A = ga^2$ and the Schrödinger-like potential $U(t)$ becomes a superposition of two potentials of the form of the kink case in section 2 (see e.g. 2.6) very far away one from each other, in a constant background.

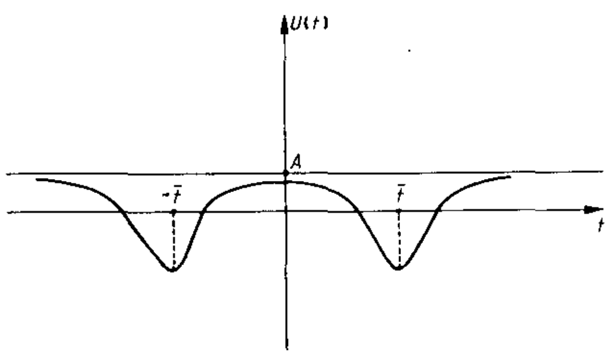

Figure 2. The shape of the potential $U(t)$ associated with the negative eigenvalue problem. $\pm\bar{t}$ are the minima of $U(t)$ and A its asymptotic value for $|t| \to \infty$.

From the expression of $U(t)$ and its form (see figure 2) we notice that we have a tunnelling effect between the two wells, such that the eigenenergy that we would have if there would exist only one well is split into two levels. In the limit $J \to 0^+$ we notice that one well is exactly equal to the potential of the kink Schrödinger-like operator (section 2), whose lowest eigenvalue is zero.

Let us call H the Schrödinger-like operator that appears in the exponent of (3.4),

$$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + U(t). \tag{3.8}$$

Since the Hamiltonian (3.8) is obtained by using the expansion around the bounce solution, its (unnormalized) eigenfunction corresponding to the zero eigenvalue is given by

$$\psi_0(t)=\frac{\mathrm{d}\bar{x}_{\mathrm{B}}(t)}{\mathrm{d}t}=\operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t-\bar{t})\right)-\operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t+\bar{t})\right). \tag{3.9}$$

However, from the shape of $\psi_0(t)$ depicted in (figure 3) we see that $\psi_0(t)$ *is not* the eigenfunction corresponding to the lowest eigenvalue, which means that there exists another eigenfunction $\psi_{-1}(t)$ corresponding to a negative eigenvalue. This eigenstate plays the role of the vacuum in the equivalent Schrödinger problem considered here. The importance of the presence of the negative eigenvalue of (3.8) is because it implies that we cannot keep only the quadratic contribution in (3.4), forcing us to devise ways to obtain information from the quartic and cubic terms [7].

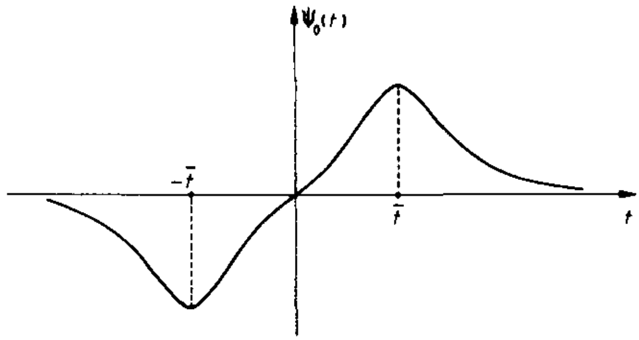

Figure 3. The shape of the zero eigenfunction of the Schrödinger equivalent problem used to take into account the quantum contribution around the classical bounce solution.

To obtain the vacuum eigenvalue of H, call it E_{-1}, we use the method for calculating the splitting of an energy level due to the existence of a barrier between two potential wells [8]. We remark that the set of the eigenfunctions and eigenvalues of the operator H_0 (2.6) is known and, in particular, the normalized zero-mode eigenfunction is

$$\varphi_0(t)=\left(\frac{3}{2\sqrt{2A}}\right)^{1/2}\operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t-\bar{t})\right). \tag{3.10}$$

Now, from the Schrödinger equations associated to the kink, $H_0\varphi_0=0$, and to the bounce solution, $H\psi_{-1}=E_{-1}\psi_{-1}$, we have, rewriting H_0 in the form $H_0=-\frac{1}{2}\,\mathrm{d}^2/\mathrm{d}t^2+V_0(t)$,

$$-\tfrac{1}{2}\int_{-\infty}^{+\infty}\psi_{-1}(t)\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}t^2}\,\mathrm{d}t+\int_{-\infty}^{+\infty}\psi_{-1}(t)V_0(t)\varphi_0(t)\,\mathrm{d}t=0 \tag{3.11a}$$

$$-\tfrac{1}{2}\int_{-\infty}^{+\infty}\varphi_0(t)\frac{\mathrm{d}^2\psi_{-1}}{\mathrm{d}t^2}\,\mathrm{d}t+\int_{-\infty}^{+\infty}\varphi_0(t)U(t)\psi_{-1}(t)\,\mathrm{d}t=E_{-1}\int_{-\infty}^{+\infty}\varphi_0(t)\psi_{-1}(t)\,\mathrm{d}t. \tag{3.11b}$$

Subtracting (3.11b) from (3.11a) and taking into account the vanishing of $\varphi_0(t)$, $\psi_{-1}(t)$ and of its derivatives at $t\to\pm\infty$, we get

$$\int_{-\infty}^{+\infty}\mathrm{d}t\,\varphi_0(t)(U(t)-V_0(t))\psi_{-1}(t)=E_{-1}\int_{-\infty}^{+\infty}\varphi_0(t)\psi_{-1}(t)\,\mathrm{d}t. \tag{3.12}$$

Up to now we have made *no* approximations in the calculations above. At this point we replace ψ_{-1} by a function that approximates it fairly well, respecting its even parity in t,

$$\psi_{-1}(t) \approx \operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t+\bar{t})\right) + \operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t-\bar{t})\right). \tag{3.13}$$

Replacing (3.13), (3.10) in (3.12), we get from (3.12) that E_{-1} is a quotient of integrals of hyperbolic functions. All the integrals in (3.12) can be exactly performed for any value of J (see appendix), but we are interested in the behaviour of E_{-1} on J (external current) for small J. After long but straightforward calculations, we obtain *up to first order in J* the value

$$E_{-1} \approx -\frac{36\sqrt{3}}{a} J. \tag{3.14}$$

4. Conclusion

We studied the splitting of the lowest energy eigenvalue of the Schrödinger equivalent problem for the bounce solution. We used the Landau-Lifshitz approach [8] to take into account the tunnelling effect. In our case we were able to get the lowest eigenvalue of H, instead of just the value of the splitting of the energy. The reason is that the time translation invariance of the classical solution forces the Schrödinger equivalent problem to have a zero eigenvalue. We showed also that E_{-1} goes to zero as $J \to 0$, as it should since the barrier between the wells tends to infinity.

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Appendix. Resumé of the integrals involved

(i) $$\int_{-\infty}^{+\infty} \mathrm{d}t \, \operatorname{sech}^4\left(\sqrt{\frac{A}{4}}(t-\bar{t})\right) = \frac{4}{3}\sqrt{\frac{2}{A}}.$$

(ii) $$\int_{-\infty}^{+\infty} \mathrm{d}t \, \operatorname{sech}^4\left(\sqrt{\frac{A}{2}}(t-\bar{t})\right) \operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t+\bar{t})\right)$$

$$= \frac{16}{3}\sqrt{\frac{2}{A}}\frac{J}{J_c}\left[\frac{-4}{(1-J/J_c)^3} + \frac{6(1+2J/J_c)}{(1-J/J_c)^4} - \frac{4J/J_c}{(1-J/J_c)^4}\ln\left(\frac{J}{J_c}\right)\right.$$
$$\left. + \frac{(8J/J_c)(2+J/J_c)}{(1-J/J_c)^5}\ln\left(\frac{J}{J_c}\right) + \frac{(2J/J_c)(2+J/J_c)}{(1-J/J_c)^4}\right]$$

where $J_c = 2ga^3/3\sqrt{3}$.

(iii) $$\int_{-\infty}^{+\infty} \mathrm{d}t\, \operatorname{sech}^4\left(\sqrt{\frac{A}{2}}(t-\bar{t})\right) \operatorname{sech}^2\left(\sqrt{\frac{A}{2}}(t+\bar{t})\right)$$
$$= -8\sqrt{\frac{2}{A}}\,\frac{J}{J_c}\left(\frac{2}{2J/J_c-1}+\frac{(1+J/J_c)^2}{1-2J/J_c}\ln\frac{J}{J_c}\right).$$

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